Topological Observations on Multiplicative Additive Linear Logic

André Hirschowitz
Université de Nice - Sophia Antipolis

Michel Hirschowitz
CEA - LIST

Tom Hirschowitz
CNRS, Université de Savoie

Abstract

As an attempt to uncover the topological nature of composition of strategies in game semantics, we present a "topological" game for Multiplicative Additive Linear Logic without propositional variables, including cut moves. We recast the notion of (winning) strategy and the question of cut elimination in this context, and prove a cut elimination theorem. Finally, we prove soundness and completeness. The topology plays a crucial role, in particular through the fact that strategies form a sheaf.

1. Overview

The notion of a game between two players (P and O) has become fundamental in proof theory and programming language theory. A natural way to think of such a game is as a directed graph, whose edges represent moves between positions, together with some information about who plays the moves.

Game semantics (Abramsky 1997; Hyland 1997) has widened this notion of game, by providing means to connect two such games together. In game semantics, each player takes part in two distinct games, and acts as P in one and as O in the other. Connection, or interaction, then happens by letting two players respectively play P and O on a common game.

By making several such connections, one obtains a sequence of games, subject to topological considerations. For example, one may see the involved games as edges in a graph with the players as vertices, as in

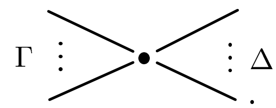
$$\xrightarrow{\text{game } 0} \text{ player } 1 \xrightarrow{\text{game } 1} \text{ player } 2 \xrightarrow{\text{game } 2} \text{ etc.},$$

and decree that an open neighborhood of player i is the sequence

$$\xrightarrow{\text{game } i-1} \text{ player } i \xrightarrow{\text{game } i} .$$

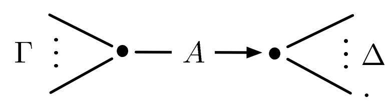
The topology here is simplistic, but arguably, this is only due to the requirement that game semantics be categorical, i.e., each player sees only two games. This is most striking in the game semantics of sequent calculi, where sequents $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ are interpreted as games $A_1 \wedge \cdots \wedge A_n \to B_1 \vee \cdots \vee B_m$.

Let us instead allow each player to see more than two games, i.e., lie in an open neighborhood like



(1)

We thus consider positions to be spaces obtained by plugging such atomic neighborhoods together. A move now leads from a position to another, where a move – in the old sense – has been played on one of the connections. We investigate this paradigm in the context of Multiplicative Additive Linear Logic without propositional variables (henceforth MALL), where logical rules, i.e., moves, are (slightly enriched) continuous functions between positions. Most emblematic is perhaps our *cut* move leading from position (1) to



(2)

It is formalised from the obvious continuous function from (2) to (1).

We investigate a few topological constructions and properties in this setting, among which:

- Strategies, defined in a suitably local way, form a sheaf. Furthermore, winning strategies are a subsheaf of strategies, i.e., the amalgamation of winning strategies is winning again.
- There is a notion of cut elimination: building upon a factorisation system, we define a construction of a cut free strategy from a strategy with cuts, again preserving the winning character.

These observations lead in the case of our semantics for MALL to standard logical results like:

Coherence There is no winning strategy on the sequent with no formula.

Correctness Any provable MALL sequent admits a winning strategy.

Completeness Any sequent with a winning strategy is provable in MALL.

2. A game for MALL

2.1 Hypersequents

As explained above, our positions have a particular structure, which we now define. First, define MALL formulae by the grammar

$$\begin{array}{rcl} A, B, C, \ldots \in \mathcal{P} & ::= & \mathbf{0} \mid \mathbf{1} \mid A \otimes B \mid A \oplus B \\ & \mid & \top \mid \bot \mid A \parr B \mid A \,\&\, B, \end{array}$$

and decree that formulae on the first line are positive, while the others negative. De Morgan duality is defined as usual (sending a connective to that vertically opposed to it). Recall in passing the corresponding sequent calculus (Girard 1987).

Say that a *partial* directed graph is a directed graph

$$E \underset{t}{\overset{s}{\rightrightarrows}} V \qquad (3)$$

with source and target maps s and t *partial*, i.e., edges may be dangling. We call edges with no source *inputs*, and dually edges with no target *outputs*.

Definition 1. *A* hypersequent *is a finite, partial directed graph, which is furthermore topologically acyclic, i.e., which is acyclic as an undirected partial graph.*

Following the intuitions in Section 1, we slightly abusively identify sequents with connected, one-vertex hypersequents as in (1).

We then endow hypersequents (3) with a topology on the coproduct $E + V$ by decreeing that a set of points is open when for each vertex, it contains all the adjacent edges. Using this topology, we build a category of hypersquents by defining a morphism $U \to V$ to be given by a continuous function from U to V as topological spaces, sending vertices to vertices. Such functions compose in the obvious way.

Remark 1 (Topology). *Observe that this entails:*

- *a set of points is closed iff for each edge it contains all the adjacent vertices,*
- *each vertex in V is a closed point,*
- *each edge in E is an open point,*
- *each edge $e \in E$ adjacent to some $v \in V$ has this v in its adherence.*

Remark 2 (Morphisms). *Morphisms are a bit like morphisms of graphs, in the sense that by continuity if an edge e adjacent to some vertex v is sent to an edge e', then the image of v is adjacent to e'. However, they differ from morphisms of graphs in that:*

- *they may reverse the direction of edges,*
- *they may sent edges to sequents, as will for example the cut move. Such edges are* collapsed *by the morphism, while the other are* persistent.

To build our category of hypersequents, we define the following generic way of labeling them. Assume given a category $\mathcal{C}$ with a polarity (positive or negative) on morphisms, such that the usual sign rules are respected by composition, e.g., identities are positive, composing two negative morphisms yields a positive one, etc. Define the category $\mathsf{G}(\mathcal{C})$ of $\mathcal{C}$-hypersequents to have

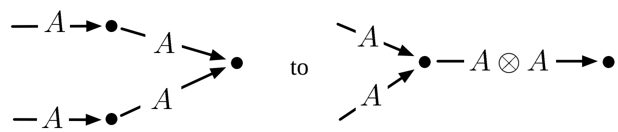
- objects: hypersequents with edges labeled in $\mathsf{Ob}(\mathcal{C})$, i.e., equipped with a function $\ell : E \to \mathsf{Ob}(\mathcal{C})$;
- morphisms $U \to V$: pairs (g, o) of a morphism $g : U \to V$ of unlabeled hypersequents, and for each persistent edge e, a morphism $o_e : \ell_U(e) \to \ell_V(g(e))$ in $\mathcal{C}$, such that if o_e is positive then the direction of e is preserved by g, and otherwise it is reversed[1].

Morphisms compose, and the condition on the direction of edges is preserved thanks to the sign rules.

We apply this construction to the category Occ with objects the positive formulae and morphisms $A \to B$ the *occurrences*, i.e., paths from the root in B reaching a subformula equal to A up to de Morgan duality. The sign of a morphism is that of the subformula reached by the path. This gives us the category $\mathsf{G} = \mathsf{G}(Occ)$.

[1] Here by direction we mean the pair (se, te), seeing s and t as functions $E \to (V + 1)$. An edge without source or target may thus have its direction both preserved and reversed.

Before going on to define the moves of our game, we show a few example morphisms. From the obvious continuous function from (the underlying space of)



we may define four different morphisms, according to the occurrences we assign to the two premises of the tensor. For example, we may send both edges to the first premise by assigning them both the occurrence 0. We also may assign the upper edge the occurrence 1 and to the lower edge the occurrence 0. There are two symmetric morphisms.

To illustrate the conventions on signs of formulae, consider the morphism from

$$\bullet \longleftarrow B^{\perp} \otimes C^{\perp} \text{———} \bullet$$

to

$$\bullet - A \otimes ((B \parr C) \parr D) \longrightarrow \bullet\ .$$

It assigns occurrence 10 to the unique edge of the domain. But since the corresponding subformula of $A \otimes ((B \,\Gamma\, C) \,\Gamma\, D)$ is negative, the edge's source and target are swapped, and the formula is dualised. Of course, we immediately introduce the notation consisting of labeling edges with negative formulae to denote the reversed edge with the dual formula. In this way, the domain of the above morphism becomes

$$\bullet \text{———} B \parr C \longrightarrow \bullet\ .$$

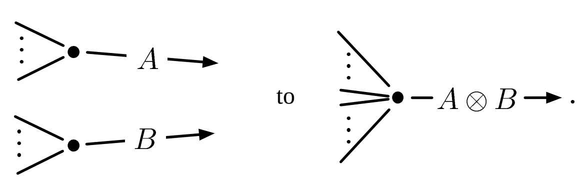
(We could also have used an equivalent category where labels may directly be negative.)

2.2 Moves

In the category G of hypersequents, we now single out a class of morphisms as our *proper moves*, thus forming a subgraph M of G. We will first define a set of *basic moves* corresponding to the rules of MALL, and then extend them by embedding.

Our basic moves are defined in Figure 1. Each line defines a move, the first being the already mentioned *cut* move. In each case, the move is the obvious morphism from left to right, the dots meaning that the move is a morphism on a larger hypersequent, which is an isomorphism outside the shown part.

Since we want to get topological, it seems natural to consider restrictions of basic moves. For example, the restriction of the tensor move to the left-hand sequent would send



To formalise this idea, we consider the identity-on-objects subcategory $\mathsf{H} \hookrightarrow \mathsf{G}$ with the same objects, and morphisms the pairs (g, o) with g an open embedding and o the function assigning to each edge labeled A the identity occurrence id_A. In the following, we call these morphisms simply *embeddings*. Observe that G has pullbacks along embeddings, that pullbacks of embeddings are embeddings again.

We can now extend our basic moves under the following rule: if a morphism m as above is the restriction of a basic move m' along

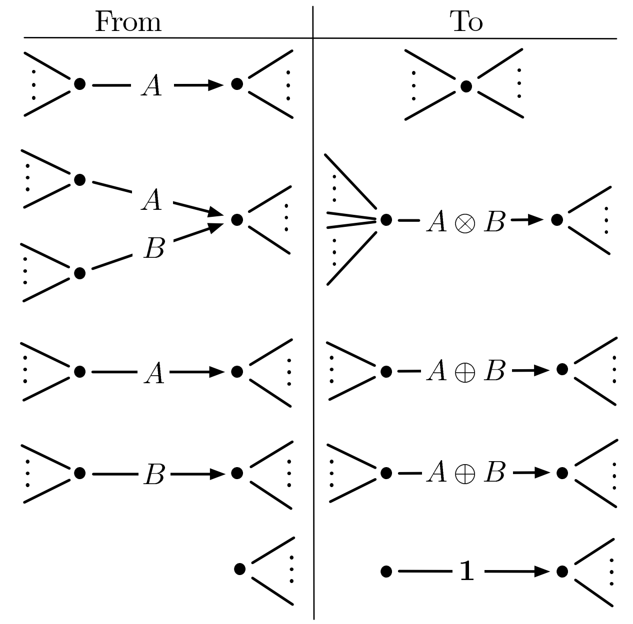


Figure 1. Basic moves

an embedding j, in a pullback square

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner" very near start] & U' \arrow[d, "m'"] \\ V \arrow[r, hook, "j"] & V', \end{tikzcd} \tag{4}$$

and if further m is not an isomorphism, then m is a *proper move.*

Finally, a vertex v, is *active* in a proper move m when either

- m is a cut and v is the cut sequent, or
- m is not a cut and v is the source of the broken edge.

There is at most one active vertex in a proper move, and we call sequents and proper moves active when they contain an active vertex, and *passive* otherwise.

2.3 Plays and strategies

To sum up, we have a site G of hypersequents, with

- an identity-on-objects subcategory $\mathsf{H} \hookrightarrow \mathsf{G}$ of embeddings,
- an identity-on-objects subgraph $\mathsf{M} \hookrightarrow \mathsf{G}$ of proper moves, stable under composition with isomorphisms,

such that

- embeddings have pullbacks in G, and these pullbacks are embeddings again,
- the pullbacks of proper moves along embeddings thus exist, and are either proper moves again, or isomorphisms.

We also have a polarity on proper moves, i.e., a partition of proper moves into passive and active ones.

Let us now define plays in this setting. Traditionally, plays are defined as sequences of moves. Here, because of the topological nature of positions, we find it useful to generalise this as follows. Consider the graph M^0 of *moves* defined by the following pushout of graphs

$$\begin{tikzcd} \mathrm{Ob}(\mathsf{G}) \arrow[r, hook] \arrow[d, hook] & \mathsf{H} \arrow[d, hook] \\ \mathsf{M} \arrow[r, hook] \arrow[ur, phantom, "\ulcorner" very near end] & \mathsf{M}^0. \end{tikzcd}$$

It has

- vertices the objects of G, and
- edges the coproduct of proper moves and embeddings.

A *play* on some object U is a path to U in M^0; it is *proper* when it has no embeddings. Let P^0 be the free category generated by M^0. Composition defines a functor $G : \mathsf{P}^0 \to \mathsf{G}$, which leave implicit except where necessary.

Let us now turn to strategies. Traditionally, strategies are non-empty, prefix-closed sets of (proper) plays. Here, we are in a topological setting, so instead of defining strategies as sets of plays, we want to include in them as local an information as possible. What strategies have to contain is, at each stage in the course of the play, for each involved edge or sequent, the moves it accepts. We formally define them to contain this information and not more. Still, (winning) strategies generate meaningful sets of plays, as we explain in a bit more detail in Section 4.1.

Call a hypersequent *atomic* when it is either empty, or an edge, or a sequent. A *thread* on a hypersequent U is a play p such that:

(T) For all proper moves $m : W \to V$ appearing in p, V is atomic.

Now, call a move $W \xrightarrow{f} V$ *mandatory* when either

- f is an embedding, or
- V is atomic and f is a passive proper move.

A *strategy* on U is then a set of threads S which is:

S1 prefix-closed, i.e., if $tt' \in S$, then also $t \in S$,

S2 stable under extension by mandatory moves, i.e., if $W \xrightarrow{f} V$ is mandatory and $V \xrightarrow{t} U$ is in S, then also tf is in S,

S3 stable under isomorphism, i.e., if for any threads

$$t' : U \to X \qquad \text{and} \qquad t : Y \to V,$$

and commuting square

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "m"'] & X' \arrow[d, "m'"] \\ Y & Y' \arrow[l, "i"] \end{tikzcd}$$

with m and m' moves and i and j isomorphisms, $tmt' \in S$ iff $tim'jt' \in S$;

S4 and stable under composition and decomposition of embeddings, i.e., for any t, t' as above and any embeddings

$$X \xhookrightarrow{h'} Z \xhookrightarrow{h} Y,$$

$t \circ h \circ h' \circ t' \in S$ iff $t \circ G(hh') \circ t' \in S$

Observe that these axioms entail the "one-sided" versions of **S3**: if, e.g., i is the identity, then $tmt' \in S$ iff $tm'jt' \in S$. Indeed, we apply **S3** twice with the squares

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "m"'] & X' \arrow[r, equal] \arrow[d, "m'"] & X' \arrow[d, "m'"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

to deduce that $t \circ id \circ id \circ m' \circ id \circ j \circ t'$ is in S, then apply **S4** with $id \circ id$, and apply **S3** again with the right-hand square above, to obtain that $tm'jt'$ is in S. The converse implication is similar.

The *restriction* $t^*(S)$ of a set S of threads on U along some thread $t : V \to U$ is the set of threads t' on V such that $tt' \in S$. The restriction $t^*(S)$ of a strategy along any t is obviously a strategy again, although possibly the empty one. (Observe in passing that a strategy may be empty.) We furthermore have, thanks to **S4**, for the obvious Grothendieck topology on H,

Theorem 1. *Strategies form a sheaf* $\mathsf{S} : \mathsf{H}^{op} \to \mathsf{Set}$.

Proof. A strategy S on U is determined by its set of restrictions $h^*(S)$ for $h : V \hookrightarrow U$ and V atomic. But any covering sieve on U includes those h's and thus entirely determines S. So the presheaf S is separated (amalgamations, when they exist, are unique). Now given a sieve $\mathcal{S}$ on U with compatible strategies S_i on the embeddings $h_i : U_i \hookrightarrow U$ of $\mathcal{S}$, define the amalgamation P as follows. First any sequence of embeddings to U is in P. Furthermore, for any thread p on U decomposing as

$$W \xrightarrow{q} V \xhookrightarrow{r} U$$

with r a sequence of embeddings and V atomic, then let U_i be one of the members of $\mathcal{S}$ isomorphic to V, and $q' : W \to U_i$ be the play corresponding to q there (recall that proper moves are stable under isomorphism). Then decree that $p \in P$ iff $q' \in S_i$. ☐

3. Cut elimination

In this section, we define our cut elimination (= descent) procedure for strategies. We start by specifying cut elimination as a function from strategies to cut free strategies: Consider the subsheaf $\mathsf{S}_{cf} \hookrightarrow \mathsf{S}$ of strategies consisting of cut free strategies, i.e., those whose plays are all in the free category P^0_{cf} generated by non cut moves. Cut elimination should provide a morphism of sheaves $ce : \mathsf{S} \to \mathsf{S}_{cf}$, preserving the winning character of strategies. In this section we stick to defining our morphism of sheaves, and defer to Section 4 the study of winning strategies.

3.1 Overview

Remote view: an easy task We will construct our morphism of sheaves using a more general family of functions $\mathsf{Desc}_c : \mathsf{S}(U) \to \mathsf{S}_{cf}(V)$ indexed by a particular class of morphisms $U \xrightarrow{c} V$ in G. The subfamily of functions $\mathsf{S}(U) \to \mathsf{S}_{cf}(U)$ obtained by taking $c = id_U$ will lead to the desired morphism of sheaves.

The involved class of morphisms c is that of *cut only topological plays*, i.e., morphisms c as above admitting a decomposition into cut moves. For each such morphism, we will define functions $\mathsf{Desc}_c : \mathsf{S}(U) \to \mathsf{S}_{cf}(V)$ sending strategies on U to cut free strategies.

The rough idea for defining these functions is natural: compute cut elimination for moves and extend it to plays by induction. Cut elimination for moves arises from a factorisation system: a morphism in G may always be decomposed into a "cut-like" morphism, followed by a "non cut-like" morphism, which yields a factorisation system $(\mathcal{L}, \mathcal{R})$. In particular, cut only topological plays are in $\mathcal{L}$, but $\mathcal{L}$ contains other morphisms, as we shall shortly see.

Given a move $W \xrightarrow{m} U$, factorisation yields the dashed arrows in

$$\begin{tikzcd} W \arrow[r, dashed, "c'"] \arrow[d, "m"'] & X \arrow[d, dashed, "q"] \\ U \arrow[r, "c"'] & V, \end{tikzcd} \tag{5}$$

with $c' \in \mathcal{L}$ cumulating the cuts in c and m, and $q \in \mathcal{R}$. We thus take $\mathsf{Desc}_c(m) = q$, and say that m *descends* along c as q.

It turns out that there are (roughly) two relevant configurations here:

- q is a move, or
- q is an identity.

We interpret the second case by saying that Desc_c should really send moves to *plays*. If q is a move, then $\mathsf{Desc}_c(m)$ is the one-move play q. Otherwise, $\mathsf{Desc}_c(m)$ is the empty play, and we replace the above square by a triangle

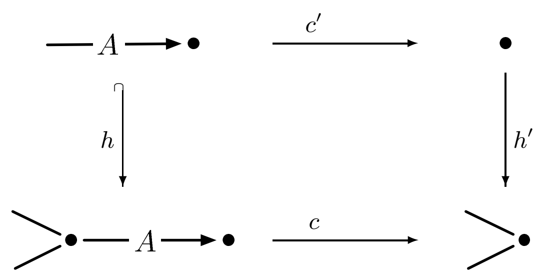
$$\begin{tikzcd} W \arrow[d, "m"'] \arrow[dr, dashed, "c'"] & \\ U \arrow[r, "c"'] & V. \end{tikzcd}$$

Cut elimination (= descent) for plays is then obtained by piling such squares and triangles: given a play $W \xrightarrow{p} U$, this yields the dashed arrows in

$$\begin{tikzcd} W \arrow[r, dashed, "c'"] \arrow[d, "p"'] & X \arrow[d, dashed, "q'"] \\ U \arrow[r, "c"'] & V, \end{tikzcd} \tag{6}$$

where q' is the concatenation of the plays obtained as above, for each move of p.

Cut elimination is partial This should define descent for plays, but things turn out to be a little more complicated, because the function Desc_c is actually *partial*. Indeed, some embeddings cause trouble, as shown by the following example. The following is a factorisation square:



Indeed, the lower-left composite may be decomposed as

- a collapse of the A-labeled edge,
- an injection of the resulting vertex into the codomain.

We cannot consider this a successful descent, for two reasons:

- c', although in $\mathcal{L}$, is no cut only topological play – cuts only collapse two-ended edges, and
- h', although in $\mathcal{R}$, is not open – since its image is not.

So our function is partial. Worse, it is much likely to be undefined on threads, which very often behave as h above, i.e., restrict to one end of an edge collapsed by c. Even worse, threads may restrict to one end of an edge created by some cut move earlier in the thread.

We thus cannot reasonably define descent for strategies as a direct extension of descent for plays, i.e., by taking $\mathsf{Desc}_c(S)$ to be the image of S under $elim$ (for some strategy S).

But, we may delineate the problem better: in a play p as in (6), call an edge *doomed* when it is collapsed by the composite cp (we indeed want doomed edges to disappear through cut elimination). At such a stage p, observe that partiality is only caused by embeddings cutting off doomed edges. We thus adapt the notion of thread

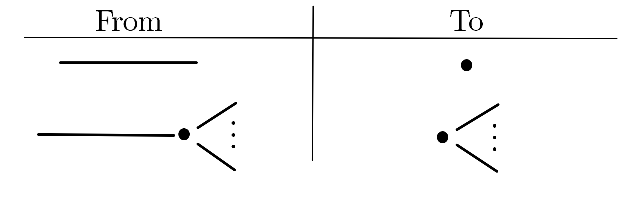


Figure 2. New steps

to a context where doomed edges are considered unbreakable. This leads to the following notion of c-*cable*.

Characterising the plays descending to threads Before playing, threads restrict to atomic hypersequents, which we now view as connected subspaces with no two-ended edge. If we now consider, at each stage, doomed edges as unbreakable, the atomic hypersequents should now be the connected hypersequents where

$$\text{doomed} \Leftrightarrow \text{two-ended},$$

i.e., a doomed arrow has two ends, and a non-doomed arrow does not. Our c-cables are thus the plays which, before playing a proper move, restrict to such subspaces. In short: before playing, cables must cut off all the edges that may be cut off. Observe that if there are no doomed edges, one exactly recovers threads.

We then may define the descent $elim_c(S)$ of a strategy S to be $\mathsf{Desc}_c(cables_c(S))$, i.e., the image by Desc_c of its c-cables.

Finalisation Then we are almost done. Beyond being partial, our function Desc_c was actually only defined up to isomorphism, as is factorisation. We thus define it as a relation, but the construction remains essentially the same. Finally, the cut elimination of S is a set of threads, but need not be a strategy, and we need to close off by Axiom **S4** to obtain one. To explain why this is so, recall that Axiom **S4** requires strategies to contain plays regardless of composition of embeddings, i.e., $G(h \circ h')$ is not distinguished from the sequence $h \circ h'$. Now, consider a descent like

$$\begin{tikzcd}
U \arrow[r] \arrow[d, hook] & U' \arrow[dd, hook, "h'"] \\
V \arrow[d, "m"'] \arrow[dr] & \\
W \arrow[r] \arrow[d, hook] & Z \arrow[d, hook, "h"] \\
X \arrow[r] & X'
\end{tikzcd}$$

as above, where some proper move m descends to the identity. The composite $G(h \circ h')$ need not be the image of any play on X. The other direction of **S4** is satisfied though, so we need only close under composition of embeddings, defining the descent of a strategy S to be the corresponding closure $ce_c(S) = \overline{elim_c(S)}$.

3.2 Factorisation

Let us start with the announced factorisation system. Given a morphism $U \xrightarrow{f} V$, we may decompose it as

$$U \xrightarrow{g} W \xrightarrow{h} V,$$

where all g does is collapse edges to vertices. Formally, g belongs to the class $\mathcal{L}$ of morphisms which may be decomposed into a sequence using only cut moves and the morphisms shown in Figure 2, plus isomorphisms.

Now, what will h look like? Obviously, h will send edges to edges. And this turns out to be enough: calling $\mathcal{R}$ the class of morphisms sending edges to edges, we have

Lemma 1. *The classes $\mathcal{L}$ and $\mathcal{R}$ form a factorisation system for* G.

To prove this, we first observe that morphisms $U \xrightarrow{c} V$ in $\mathcal{L}$ are epi. Indeed, their underlying functions are surjective, and moreover, for edges of V, precomposition by c does not change the occurrences. This leads to

Proof of Lemma 1. Existence of a factorisation is obvious. Now, consider a commuting square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "c"'] & Y \arrow[d, "r"] \\
U \arrow[r, "g"] & V
\end{tikzcd} \tag{7}$$

with $c \in \mathcal{L}$ and $r \in \mathcal{R}$. Then choose a factorisation (c', r') for f, as in

$$\begin{tikzcd}
& W \arrow[dr, "r'"] & \\
X \arrow[ur, "c'"] \arrow[rr, "f"'] & & Y.
\end{tikzcd}$$

Now, for any morphism $X \xrightarrow{f} Y$, let $col(f)$ be the set of edges in X collapsed by f, i.e., sent to vertices.

We have

$$col(c) \subseteq col(f) = col(c').$$

By collapsing exactly the edges in $c(col(c'))$, we define a morphism c'' such that

$$\begin{tikzcd}
& U \arrow[dr, "c''"] & \\
X \arrow[ur, "c"] \arrow[rr, "c'"'] & & W.
\end{tikzcd}$$

All in all, we obtain a diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "c'"] \arrow[dd, "c"'] & & Y \arrow[dd, "r"] \\
& W \arrow[ur, "r'"] \arrow[dr, phantom, "?"] & \\
U \arrow[ur, "c''"] \arrow[rr, "g"] & & V,
\end{tikzcd}$$

where the upper triangles and the perimeter are known to commute. But a simple diagram chase shows that c equalises the lower triangle, i.e., $gc = rr'c''c$. But c is epi, so the lower triangle commutes.

This yields a diagonal for the original square (7), making both triangles commute. Its uniqueness is a direct consequence of c being epi. □

3.3 The partial "function"

We then define our relation on plays (which is more like a partial function up to isomorphism), defined as a bipartite graph Desc: when is a cut free play the cut elimination of a given play? In order to define this, we start with the corresponding relation on moves, not trying for the moment to understand which moves have an image. Consider the graph C^0 with vertices the cut only topological plays $c : U \to V$, and two kinds of edges $c \to c'$, based on the squares

$$\begin{tikzcd} U' \arrow[r, "c'"] \arrow[d, "m"] & V' \arrow[d, "m'"'] \\ U \arrow[r, "c"] & V \end{tikzcd} \quad \begin{tikzcd} U' \arrow[r, "c'"] \arrow[d, hook, "h"] & V' \arrow[d, hook, "h'"'] \\ U \arrow[r, "c"] & V \end{tikzcd} \quad \begin{tikzcd} U' \arrow[r, "c'"] \arrow[d, "m"] & V' \arrow[d, equal] \\ U \arrow[r, "c"] & V, \end{tikzcd}$$

where

- the right-up sequence is a $(\mathcal{L}, \mathcal{R})$-factorisation of the left-low composite,
- m and m' are proper moves, with m' non cut,
- h and h' are embeddings,
- c and c' are cut only topological plays.

We define our graph C^0 to have as edges $c \to c'$ the squares as the first two above, and the triangle

$$\begin{tikzcd} U' \arrow[d, "m"'] \arrow[dr, "c'"] & \\ U \arrow[r, "c"] & V \end{tikzcd}$$

for each square as the third one above. This graph freely generates a category C^1 whose morphisms $c \to c'$ are piles of such squares and triangles. Taking the left- and right-hand sides of such piles yields source and target functors to the category P^0 of plays. However, we make the distinction with the category P^0_{cf} of cut free plays, and denote by

$$\mathsf{P}^0 \xleftarrow{s} \mathsf{C}^1 \xrightarrow{t} \mathsf{P}^0_{cf}$$

the corresponding source and target functors. This defines a bipartite graph Desc between plays and cut free plays, and we say that p *descends* along c as q when there is an edge $p \to q$ in Desc with lower border c.

This notion of descent extends by union to a function on sets of plays S: $\mathsf{Desc}_c S$ is the set of plays descending from plays in S along c.

However, this does not meaningfully send strategies to strategies, because threads do not in general descend along such c's. Nor can we prove for free that it preserves the winning character: the image of a given strategy could *a priori* be empty.

So, in the next section, we start investigating conditions for moves and plays to descend along a given c in Section 3.4. This leads to the notion of c-compatibility. Using this, we define our c-cables in Section 3.5, which all descend to threads along c. We then turn back to strategies, and after defining (Section 3.6) the plays $\hat{S}$ generated by a strategy S over V, we define the descent of S along c to be the set of threads over V descending from a play in $\hat{S}$. However, the result of descent need not be a strategy, and we still must close under (one direction of) axiom **S4**. We show in Section 4 that this notion of descent preserves the winning character (as defined there) of strategies. Cut elimination is recovered as the special case where c is the identity.

3.4 When plays descend: compatibility

We now turn to characterising moves that descend along a given c. We further give a sufficient condition for descending plays: c-compatibility.

Characterising cut only topological plays Recall that a *topological play* is any map in G which may be decomposed into moves. Further call *cut only* any morphism in $\mathcal{L}$. A cut only morphism does not have to be a topological play, as shown for example by the morphisms in Figure 2. Indeed, cut moves only collapse two-ended edges. However, given any morphism $c : U \to V$, the following are easily shown equivalent:

(i) c is a topological play and a cut only morphism,

(ii) c is cut only, and it only collapses two-ended edges,

(iii) c admits a decomposition into cut moves.

Hence, it is consistent to take as we did *cut only topological play* to mean topological play admitting a decomposition into cut moves.

Proper moves We start by proving that everything goes smoothly for proper moves.

Lemma 2. *If $U \xrightarrow{m} V$ is a proper move and e is a two-ended edge in V, then the edges in $m^{-1}(e)$ are also two-ended in U.*

Proof. By case inspection this holds for basic moves, and it remains true after any restriction. □

Recall that $col(p)$ denotes the set of edges in U collapsed by p, i.e., sent to vertices, for any $U \xrightarrow{p} V$.

Lemma 3. *Any sequence $W \xrightarrow{m} V \xrightarrow{c} U$ with m any proper move and c a cut only topological play may be completed as a commuting square*

$$\begin{tikzcd} W \arrow[r, "c'"] \arrow[d, "m"'] & W' \arrow[d, "f"] \\ V \arrow[r, "c"] & U \end{tikzcd} \tag{8}$$

in G, with f an isomorphism or a cut free proper move and c' a cut only topological play. Furthermore, $col(cm) = col(c')$.

Proof. Let E^c_V be the set of edges collapsed by c, and E^c_W their antecedents by m. By Lemma 2, the edges in E^c_W are two-ended.

If m is a cut move, then let E^m_W be the set of edges collapsed by m. We have by construction E^c_W and E^m_W disjoint. Let $c' : W \to W'$ be the cut only map obtained by collapsing exactly these edges $E_W = (E^c_W \uplus E^m_W)$ in W. Since the edges in E_W have two ends, c' is a topological play, and we have an isomorphism $f : W' \cong U$.

If m is not a cut move, let $c' : W \to W'$ be the cut only topological play collapsing exactly E^c_W. It remains to find f as in (8). For this, let E^m_W be the set of edges in W which are not assigned the empty occurrence by m, i.e., which are acted upon by m. All such edges are sent to a unique edge e_0 in V, and being in E^c_W for $e \in E^m_W$ is the same as being in E^c_V for e_0. Thus, we have either $E^m_W \subseteq E^c_W$, or E^m_W disjoint from E^c_W.

Now, if $E^m_W \subseteq E^c_W$, we again have an isomorphism $f : W' \cong U$, and we are done. Otherwise, E^m_W is left untouched by c', and we may mimic the action of m on the image of E^m_W by c', and land in U, making the square (8) commute in G.

In all cases, clearly, $col(cm) = col(c')$. □

Embeddings We have seen in Section 3.1 that this does not work for embeddings in general. However, the process works smoothly when such an h does not cut off any edge collapsed by c. Formally:

Lemma 4. *For any square*

$$\begin{tikzcd} U' \arrow[r, "c'"] \arrow[d, hook, "h"'] & V' \arrow[d, "h'"] \\ U \arrow[r, "c"] & V \end{tikzcd} \tag{9}$$

with h an embedding, c a cut only topological play, and (c', h') an $(\mathcal{L}, \mathcal{R})$-factorisation of ch, the following are equivalent

(i) *h' is an embedding and c' is a cut only topological play,*
(ii) *any edge $e \in col(ch)$ has two ends in U',*
(iii) *any two-ended edge $e \in col(c)$ in the image of h has a two-ended antecedent in U'.*

Furthermore, in this case $col(ch) = col(c')$ and the square is a pullback.

We first prove two easy lemmas:

Lemma 5. *Consider a diagram $U' \overset{h}{\hookrightarrow} U \xrightarrow{c} V$ with h an embedding, c a cut only topological play, and such that any edge $e \in col(ch)$ has two ends in U'. For any vertex $v_U \in U$ in the image of h, all of $c^{-1}(c(v_U))$ is in the image of h too.*

Proof. Let $v_V = c(v_U)$. Since c is a play, $c^{-1}(v_V)$ is connected and has only two-ended edges, hence is a tree in the graph-theoretical sense. But h is open, so any edge e adjacent to v_U has an antecedent by h. But since any edge $e \in col(ch)$ has two ends in U', the other end v' of e also has an antecedent. But similarly any edge incident to v' has an antecedent. By induction on the length of the path from v_U, all of $c^{-1}(v_V)$ is in the image of h. □

Here is the second lemma:

Lemma 6. *If in a triangle*

$$\begin{tikzcd} & V \arrow[dr, hook, "h'"] & \\ U \arrow[ur, hook, "h''"] \arrow[rr, hook, "h"] & & W \end{tikzcd}$$

h and h' are embeddings, then so is h''.

Proof. Obviously, h'' is injective since h is. Moreover, since h and h' have empty occurrences, h'' has empty occurrences. Finally, h'' is open: for any open $X \subseteq U$, $h''(X)$ is equal to $h'^{-1}(h(X))$, which is open since h' is continuous and h is open. □

We turn back to the proof of Lemma 4.

of Lemma 4.. First of all, $col(ch) = col(h'c') = col(c')$.

Then, *(ii)* implies *(iii)*, because any $e \in col(c)$ in the image of h is in $col(ch)$.

Conversely, *(iii)* implies *(ii)*, because given $e \in col(ch)$, $h(e)$ is in $col(c)$. But c is a play, so $h(e)$ has two ends, and so by *(iii)*, e too has two ends.

Furthermore, *(i)* implies *(ii)*, since c' is a play.

Finally, if any edge $e \in col(ch)$ has two ends in U', since c' collapses exactly the doomed edges in U' and these are all two-ended, c' is a cut only topological play.

To show that h' is open, first consider any vertex v_V in V, and any edge e_V incident to v_V. There is a unique pair (v_U, e_U) with e_U incident to v_U in U, sent to (v_V, e_V) by c. (Indeed, c leaves persistent edges untouched and does not augment their adherence.)

Now, if v_V is in the image of h', then it is in the image of $c'h'$, because c' is surjective. Moreover, since c' is a play, it has some antecedent vertex $v_{U'}$ in U'. Now, let $v'_U = h(v_{U'})$. It is sent to v_V by c, so by Lemma 5, all of $c^{-1}(v_V)$ is in the image of h. Hence v_U has an antecedent $v'_{U'}$ in U'. But since h is open, e_U also has an antecedent, left untouched by c', and hence e has an antecedent by h'. So, h' is open and *(ii)* implies *(i)*.

Finally, consider the morphism f induced by universal property of pullback in

$$\begin{tikzcd} U' \arrow[dr, dashed, "f"] \arrow[drr, bend left, "c'"] \arrow[ddr, bend right, "h"'] & & \\ & W \arrow[r, "c''"] \arrow[d, hook, "h''"] \arrow[dr, phantom, "\lrcorner", very near start] & V' \arrow[d, hook, "h'"] \\ & U \arrow[r, "c"] & V. \end{tikzcd}$$

Considering the lower-left triangle, by Lemma 6, f is an embedding. But by Lemma 5, for any vertex $v_V \in h'(V')$, all of $c^{-1}(v_V)$ is in the image of h. So, since the pullback is isomorphic to $c^{-1}(h'(V'))$, f is surjective on vertices. Now, since c' is surjective, each edge $e_V \in h'(V')$ has an antecedent $e_{U'}$ in U', hence f is surjective, hence is an isomorphism. □

In particular, when h does not cut off any two-ended edge, or equivalently when h is just a restriction to some of the connected components of V, the process works for any c. We call such h's *cut compatible*.

A sufficient condition for descending plays Using Lemmas 3 and 4, we are now able to derive the following sufficient condition by induction. For $c : U \to V$ any cut only topological play, and $W \xrightarrow{r} U$ a play, call *doomed* the edges of $col(cp)$. (In the following, we freely write "doomed in W" when r is clear from context.)

Definition 2. *A play $X \xrightarrow{p} U$ is c-*compatible *when for each decomposition of p into plays*

$$X \xrightarrow{q} W \xrightarrow{r} U, \tag{10}$$

any edge doomed in W, i.e., in $col(cr)$, has two ends.

We may characterise c-compatible plays as follows.

Lemma 7. *A play p is c-compatible iff for any decomposition*

$$X \xrightarrow{q'} Y \overset{h}{\hookrightarrow} Z \xrightarrow{r'} U \tag{11}$$

of p with h an embedding, h does not cut off doomed edges, i.e., if an edge $e \in col(cr'h)$ is such that $h(e)$ has two ends, then e has two ends in X.

Proof. Assume p has a decomposition (11) as above, but with an edge $e \in col(cr'h)$ lacking at least one end and such that $h(e)$ has two ends. Then, by taking $W = Y$, $q = q'$, and $r = r'h$, e contradicts c-compatibility of p.

Conversely, it is enough to show that for any $X \xrightarrow{p} U$ satisfying the condition, any edge in $col(cp)$ has two ends. We proceed by induction on p, using Lemma 4 for the induction step (the case of proper moves being easy). □

We have:

Lemma 8. *Any c-compatible play p descends to some cut free play along c, in a square*

$$\begin{tikzcd} X \arrow[r, "c'"] \arrow[d, "p"'] & Y \arrow[d, "p'"] \\ U \arrow[r, "c"] & V \end{tikzcd} \tag{12}$$

with p' a cut free play and c' a cut only topological play. Again, $col(cp) = col(c')$.

Proof. By induction. The induction step uses Lemmas 3, 4, and 7. □

We will now define our c-cables using c-compatibility.

3.5 Cables

Given a cut only topological play $U \xrightarrow{c} V$ as above, a play $Y \xrightarrow{r} U$ is *c-atomic* when Y is connected and its edges are doomed exactly when they have two ends.

Definition 3. *A c-cable is a c-compatible play $W \xrightarrow{p} U$ such that for any decomposition*

$$W \xrightarrow{q} X \xrightarrow{m} Y \xrightarrow{r} U$$

of p with m a proper move, r is c-atomic.

But, we have

Lemma 9. *Being atomic is equivalent to being connected and having no two-ended edge.*

Proof. Atomic hypersequents satisfy the condition. Conversely, non atomic, connected hypersequents all have at least one two-ended edge. □

This yields:

Lemma 10. *A c-compatible play $Y \xrightarrow{r} U$ is c-atomic iff in its descent square*

$$\begin{tikzcd} Y \arrow[r, "c_Y"] \arrow[d, "r"'] & Y' \arrow[d, "r'"] \\ U \arrow[r, "c"] & V, \end{tikzcd}$$

Y' is atomic.

Proof. Let E be the set of two-ended edges in Y, not in $col(c_Y)$, i.e., not in $col(cr) = col(c_Y)$. Let E' be the set of two-ended edges in Y'. Since the edges outside $col(c_Y)$ are left untouched by c_Y, E' is non-empty iff E is non-empty.

Moreover, c_Y is a topological play, so Y is connected iff Y' is connected.

By the previous Lemma, this gives the expected result. □

Finally, this entails:

Lemma 11. *Any c-cable $U' \xrightarrow{p} U$ descends as a thread.*

Proof. Cables are c-compatible, so we may consider the descent square

$$\begin{tikzcd} U' \arrow[r, "c'"] \arrow[d, "p"'] & V' \arrow[d, "p'"] \\ U \arrow[r, "c"] & V. \end{tikzcd}$$

Now, for p' to be a thread, it suffices to consider any of its decompositions as

$$V' \xrightarrow{q'} X' \xrightarrow{m'} Y' \xrightarrow{r} V$$

and show that Y' is atomic. But descent is defined inductively, so such a decomposition yields a decomposition

$$\begin{tikzcd} U' \arrow[r, "c'"] \arrow[d, "q"'] & V' \arrow[d, "q'"] \\ X \arrow[r, "c_X"] \arrow[d, "m"'] & X' \arrow[d, "m'"] \\ Y \arrow[r, "c_Y"] \arrow[d, "r"'] & Y' \arrow[d, "r'"] \\ U \arrow[r, "c"] & V \end{tikzcd}$$

of the above descent square. Because p is a c-cable, r is c-atomic, so by the previous lemma, Y' is atomic. □

We now turn to exploiting this to descend strategies. To do that, we need to carefully select the c-cables complying with a given strategy. We first define in the next section the threads $\tilde{p}$ underlying a given play p, and then define the c-cables of a strategy S to be those c-cables p such that $\tilde{p} \subseteq S$.

3.6 The threads of a play

To any play p on U, what its set of threads should be is intuitively clear, but is a bit tricky to formalise. What we do is define a graph $\mathsf{P}(U)$ of "embeddings" between plays on U. Intuitively, in this graph, an edge $p \to p'$ indicates how at each stage p sees part of what happens in p'. The set $\tilde{p}$ of threads of p will then consist of all threads t with an edge $t \to p$. This extends by union to sets of plays, so, to any strategy S on U, we may associate the set $\hat{S}$ of plays p on U whose threads are all in S, i.e., such that $\tilde{p} \subseteq S$.

It remains to define our graph $\mathsf{P}(U)$. First, consider the graph M^1 with vertices the embeddings $i : U \hookrightarrow V$ and whose edges $i \to j$ have one of the following forms

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, "m'"'] \arrow[dr, phantom, very near start, "\lrcorner"] & V \arrow[d, "m"] \\ U' \arrow[r, hook, "j"] & V' \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook, "i"] \arrow[dr, hook, "j"'] & V \arrow[d, "m"] \\ & V' \end{tikzcd}$$

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[dr, hook, "j"'] & V \arrow[d, hook, "k"] \\ & V' \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, hook, "h"'] & V \\ U', \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where the first square is a pullback and the second diagram is such that the induced square

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d, equal] \arrow[dr, phantom, very near start, "\lrcorner"] & V \arrow[d, "m"] \\ U \arrow[r, hook, "j"] & V' \end{tikzcd}$$

is a pullback, and where m and m' denote proper moves and h and k denote embeddings, all seen as moves. This graph freely generates a category P^1, whose morphisms are piles of such diagrams. Furthermore, there are morphisms of graphs $s, t : \mathsf{M}^1 \to \mathsf{P}^0$ sending the squares and triangles to their vertical borders. By adjunction, they induce functors $s, t : \mathsf{P}^1 \to \mathsf{P}^0$. This structure now induces a "horizontal" graph P whose vertices are plays, and whose edges $p \to q$ are morphisms $i \to j$ in P^1 with left-hand border p and right-hand border q.

Finally, the graph $\mathsf{P}(U)$ evoked above has vertices the plays on U, and edges the edges in P with lower border the identity. Thus, $\tilde{p}$ is the set of threads t on U such that there exists an edge $t \to p$ in P with lower border the identity.

3.7 Cut elimination

For any strategy S on U, we at last define the set $cables_c(S)$ of *c-cables* of S to be the set of c-cables p with $\tilde{p} \subseteq S$.

Recall that Desc_c sends sets of plays S to the set of plays descending from plays in S along c. We set:

Definition 4. *For any strategy S, let*

$$elim_c(S) = \mathsf{Desc}_c(cables_c(S)).$$

We then obtain:

Lemma 12. *The set of threads $elim_c(S)$ satisfies axioms **S1** to **S3** for strategies, plus one direction of axiom **S4**, namely that (in the same setting) if $t \circ G(hh') \circ t' \in S$, then $t \circ h \circ h' \circ t' \in S$.*

We first prove that the corresponding direction of **S4** holds for cables:

Lemma 13. *For any strategy S, if $t \circ G(hh') \circ t' \in cables_S()$, then $t \circ h \circ h' \circ t' \in cables_S()$.*

Proof. Being a thread of a cable is insensitive to composition or decomposition of embeddings. Indeed, any edge in the graph $\mathsf{P}(U)$ between sequences of embeddings may be obtained by piling up triangles as in

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook] \arrow[dr, hook] \arrow[ddr, hook] & U' \arrow[d, hook] \\
V \arrow[dr, hook] & V' \\
\vdots & \vdots \\
W \arrow[r, hook] & W',
\end{tikzcd}$$

with no constraint on the numbers of embeddings on each side; only the commutativity of the outer diagram matters in the end. ☐

Proof of Lemma 12. By Lemma 11, $elim_c(S)$ is a (non-empty) set of threads. Also, since S is a strategy, $elim_c(S)$ is prefix-closed. Furthermore, for any mandatory move f extending $p' \in elim_c(S)$, f easily lifts to a mandatory move in the corresponding cable, which descends as f, hence $elim_c(S)$ is stable under extension by mandatory moves. Furthermore, $elim_c(S)$ is stable under isomorphism, by construction of Desc. Finally, if $t \circ G(hh') \circ t'$ is in $elim_c(S)$, then $G(hh')$ comes from an edge in Desc, i.e., a square

$$\begin{tikzcd}
X' \arrow[r, "c_X"] \arrow[d, "h''"'] & X \arrow[d, "G(hh')"] \\
Y' \arrow[r, "c_Y"] & Y
\end{tikzcd}$$

with h'' an embedding. But by the pullback Lemma and Lemma 4, we may choose pullbacks as in

$$\begin{tikzcd}
X' \arrow[r, "c_X"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "h'"] \\
Z' \arrow[r, "c_Z"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "h"] \\
Y' \arrow[r, "c_Y"] & Y
\end{tikzcd}$$

such that $h_1 h_2 = h''$. But then by **S4** for S, we could replace h'' by $h_1 h_2$ in the cable descending to $t \circ G(hh') \circ t'$, and obtain a cable descending to $thh't'$.

☐

However, as we have seen in Section 3.1, the set of threads $elim_c(S)$ need not be closed under the other direction of **S4**. But we may perfectly close a set of threads under composition of embeddings: consider the rewriting relation on plays defined by

$$thh't' \to t \circ G(hh') \circ t,$$

and given a set of threads S, let $\overline{S}$ be the set of plays reachable from S by this relation. We have

Lemma 14. *The set of threads $\overline{elim_c(S)}$ is a strategy.*

Proof. Axioms **S1**-**S3** are preserved by $\overline{\cdot}$, as well as the first direction of Axiom **S4**. The second condition is now satisfied, hence $\overline{elim_c(S)}$ is a strategy. ☐

We may then define our family of functions: for any cut only topological play $U \xrightarrow{c} V$ and set of threads S on U, let $ce_c(S) = \overline{elim_c(S)}$, and $ce_U(S) = \overline{elim_{id_U}(S)}$. We have seen that if S is a strategy, then so is $ce_U(S)$. We further have

Lemma 15. *The functions $ce : \mathsf{S}(U) \to \mathsf{S}_{cf}(U)$ define a morphism of sheaves.*

Proof. Restriction commutes with cut elimination. ☐

4. Logic

We at last start using our game as a model of MALL. We first define winning strategies, and we relate them to more standard notions, and discuss categories of games and strategies. We then show that winning strategies are stable under cut elimination, and obtain coherence as a corollary. We then show that every MALL proof generates a winning strategy and *vice versa*, hence our model is correct and complete. (This would have entailed coherence in a less direct way.)

4.1 Winning strategies

When should a strategy be winning? Since at any stage and on any sequent it has to accept all negative moves, it reaches sequents with only $\top$'s and positive edges. In such a sequent, if there actually are some $\top$ edges, then the play should be considered won, thanks to the $\top$ axiom of MALL. Otherwise, there are only positive edges, and the strategy should propose a positive proper move. In other words, when a winning strategy is stuck, it has to be on a position with a $\top$ edge.

More formally:

Definition 5. *A sequent is* positive *when it has no input edge. A set of threads S is* winning *when it is non empty, stable under extension by mandatory moves, and when every thread in S ending on a positive sequent has an extension by a proper move in S.*

Equivalently, we may call a play $V \xrightarrow{p} U$ *maximal* in some set of threads S if it is atomic and has no extension by a proper move in S. If S is a strategy, then for such a maximal p, any negative edge of V is labeled $\top$, otherwise there is an extension by a passive proper move. Let now such a maximal play in a set of threads S be *won* when V is either empty, or a $(\top)$ edge, or a sequent with a negative edge. Otherwise it is *lost*. A position is maximal if it is the domain of a maximal thread.

Lemma 16. *A strategy is* winning *iff it is non-empty and all its maximal positions are won.*

In game theory, and in particular in game semantics, strategies are usually defined as sets of plays (without embeddings). This raises the question: in which sense is the notion of a winning strategy S related to its set of plays $\hat{S}$?

First, observe that the set of plays $\hat{S}$ is prefix-closed, and *welcoming*, in the sense that it is stable under extension by a passive proper move or an embedding. Indeed, the threads of any such extension of a play $p \in \hat{S}$ are either already threads of p, or extensions of one of them by a passive proper move or an embedding, hence again in $\hat{S}$.

Let a play p in a set of plays P be *maximal* when p has no extension by a proper move in P. Call a maximal p *won* when all its sequents have at least one negative edge. Observe that it is different to be maximal as a thread or as a play: a thread is maximal as a thread only if its domain V is atomic.

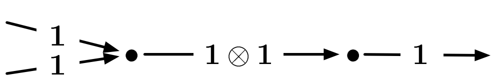
Lemma 17. *If S is a winning strategy, then any maximal play in $\hat{S}$ is won.*

Proof. Assume given a play $p : V \to U$ in $\hat{S}$ with a sequent s without any negative edge, and consider a thread t leading to it in S. If s has no edge, then t is maximal and lost, contradicting the winning character of S. Thus, s has some positive edges. But again, since S is winning, t cannot be maximal, so it has an extension by a proper move. Other sequents have to accept this move because S is winning, so V could not be maximal. ☐

All in all, we have

Theorem 2. *The set of plays $\hat{S}$ of a winning strategy is non-empty, prefix-closed, and welcoming, and its maximal plays are all won.*

However, a set of plays may satisfy the conditions of Theorem 2 without being generated by a strategy. The main reason is because these conditions miss stability under restriction and amalgamation. For instance, on the hypersequent



consider the winning set of plays P with

- proper plays choosing one repartition of the left-hand $\mathbf{1}$ edges,
- plays after restriction to the left-hand sequent choosing the other,

which satisfies the conditions, but is not generated by a strategy. Indeed, any strategy having at least the threads in $\tilde{P}$ would allow both repartitions of the left-hand $\mathbf{1}$ edges in its global plays.

4.2 Coherence, correctness, and completeness

We now turn to proving the announced logical results: coherence, correctness, and completeness. We start by proving that winning strategies are stable under cut elimination.

Theorem 3. *If S is winning, then so is $ce_c(S)$.*

Proof. First, if $elim_c(S)$ is winning, then so is $\overline{elim_c(S)}$, since no maximal positions are added. Let us thus show that $elim_c(S)$ is winning.

Let $p' : V'' \to V$ lead to a maximal position V'' in $elim_c(S)$, and choose $p : U'' \to U$ as in

$$\begin{tikzcd} U'' \arrow[r, "c''"] \arrow[d, "p"'] & V'' \arrow[d, "p'"] \\ U \arrow[r, "c"] & V \end{tikzcd}$$

maximal in $cables_c(S)$ descending to p', i.e., p has no extension in $cables_c(S)$ also descending to p'. By maximality, V'' is atomic, so by Lemma 10, p is c-atomic. So, if V'' is either an edge or empty, then so is U''. Otherwise, V'' is a sequent, so U'' is a connected hypersequent with the same one-ended edges. Now, we claim that every sequent in U'' has a negative edge. Indeed, if any sequent there had

- no edge at all, then a thread in S would lead to it, contradicting the winning character of S,
- only positive edges, then because S is a winning strategy U'' would admit an extension by a proper move in $cables_c(S)$, contradicting its maximality.

So, each sequent in U'' has at least one negative edge. But this easily implies that U'' has at least one input edge, which then has to be a $\top$. Therefore, V'' has an input $\top$ edge and is thus won. ☐

This directly entails coherence:

Corollary 1. *There is no winning strategy on the empty sequent.*

Proof. Any winning strategy S would yield a cut free one $ce(S)$. But the latter cannot be winning, as it has no proper move, so the empty sequent is maximal, but lost. ☐

4.3 Correctness and completeness

We now investigate the correspondence with provability in MALL. We defer a proof theoretical investigation to further work.

Lemma 18. *If a sequent Γ is provable in MALL, then there is a winning strategy on it.*

Proof. By standard proof technology, Γ admits a cut free proof π which at any stage starts by completely breaking negative connectives, and with no axiom links (i.e., conclude by $\mathbf{1}$ and $\top$ rules).

We proceed by induction on this π. Observe that at any stage, we must accept all embeddings. We do so implicitly, and need only specify a strategy when such embeddings lead to a sequent (on edges, a strategy has to accept all moves).

Now, let us review the base cases. If π is a $\mathbf{1}$ rule, then apply the $\mathbf{1}$ move to reach an empty position, which is hence won. If π is a $\top$ rule, then by hypothesis the only negative formulae of Γ are $\top$'s, so there are no possible passive proper moves, and Γ is maximal, hence won.

For the induction step, first, accept all passive proper moves, which in various paths lead to a sequent Γ' with only $\top$'s and positive formulae. The proof π chooses one such path to Γ'. Now, if this path is non empty, then the size has decreased, so we may apply the induction hypothesis. Otherwise, $\Gamma = \Gamma'$, and π starts with an active proper move m, reaching premisses $\pi_1, \ldots, \pi_n$, with $n \in \{1, 2\}$, which in turn have to perform negative rules to reach premisses $\pi'_1, \ldots, \pi'_n$. We choose m as the next move of our strategy. Then, accept all passive proper moves and embeddings, which (among others) lead in various paths to the conclusions of $\pi'_1, \ldots, \pi'_n$. Finally, conclude by induction hypothesis. ☐

Lemma 19. *If a sequent Γ admits a winning strategy, then it is provable in MALL.*

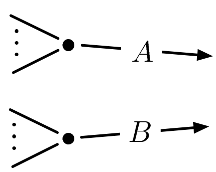
Proof. Assume given such a winning strategy S, which we may suppose cut free w.l.o.g. Since in the game, a play has a finite number of proper moves, we may take this as its size. The size of a strategy is then the maximum size of its plays.

Proceeding by induction on the size of S, if S has size 0, then because it is winning on Γ and Γ is atomic, Γ is maximal, so it has a $\top$ edge, hence is an axiom of MALL.

Otherwise, choose a thread in S performing all possible passive proper moves, and reach a sequent Γ' with only $\top$'s and positive edges. If the followed path has at least one proper move, then the size has decreased so by induction hypothesis we get a proof of Γ', to which we apply all the corresponding negative rules to get a proof of Γ. Otherwise the followed path is empty, and $\Gamma = \Gamma'$ has only $\top$'s and positive edges. If it has a $\top$, then Γ is an axiom of MALL.

Otherwise, since S is winning, there is a (active) proper move from Γ.

- If it is a $\mathbf{1}$ move, then Γ is the sequent with exactly one $\mathbf{1}$ formula, which is an axiom of MALL.
- If it is a $\oplus$ move, then it leads to some sequent Γ''. By induction hypothesis, the sequel of S being winning, we get a proof of Γ'', to which we apply the corresponding rule to get a proof of Γ.
- If it is a $\otimes$ move, then it leads to some hypersequent of the shape



that is, a disjoint union of two sequents. Since S is a strategy, we may follow the restrictions to each sequent Γ_1 and Γ_2, apply the induction hypothesis there to get proofs π_1 and π_2, to which we apply the tensor rule to get a proof of Γ.

□

We have proved:

Theorem 4. *The topological game for MALL is (logically) sound and complete.*

4.4 Towards categories of strategies

Without cut elimination, we may construct a category Strat_0 of strategies for our topological game for MALL, which we define to be the strictification of the bicategory of cospans, with

- objects the formulae, and
- morphisms $A \to B$ consisting of a cospan

$$A \hookrightarrow U \hookleftarrow B$$

in H, equipped with a strategy on U, with A and B dangling edges labeled with formulae A and B, and U a connected hypersequent with exactly one input edge – the image of A, and one output edge, the image of B.

Gluing two hypersequents along a edge which is input on one side and output on the other clearly preserves acyclicity, hence we may hope to define composition as a strategy on the (chosen) pushout. Now, observe that the unique strategy on an edge (alone) is the total one, i.e., the set of all plays. Indeed, all proper moves are passive and atomic. Thus, any two strategies $p : A \to B$ and $q : B \to C$, have the same restriction to B, hence have a unique amalgamation, which we elect to be their composition $q \circ p : A \to C$. The identity on A is given by the unique strategy on the edge labeled A. Since winning strategies are stable under amalgamation, we may form the subcategory WStrat_0 of winning strategies. We could also do the same with cut free strategies.

However, these categories Strat_0 and WStrat_0 are not quite what game semanticists are used to. Indeed, given two strategies $S : A \to B$ and $S' : B \to C$, i.e., on objects like

$$— A \rightarrow \bullet — B \rightarrow \quad \text{and} \quad — B \rightarrow \bullet — C \rightarrow$$

respectively, a game semanticist expects their composition

$$A \xrightarrow{S} B \xrightarrow{S'} C \tag{13}$$

to be a strategy on the hypersequent U:

$$— A \rightarrow \bullet — C \rightarrow, \tag{14}$$

not on V:

$$— A \rightarrow \bullet — B \rightarrow \bullet — C \rightarrow, \tag{15}$$

as in Strat_0.

Now, let $\Gamma = (A_1, \ldots, A_n)$ and $\Delta = (B_1, \ldots, B_m)$ be lists of formulae. We write $U : (\Gamma \triangleright \Delta)$ when the connected hypersequent U has exactly n input edges labeled with the A_i's and m output edges labeled with the B_j's. For any such hypersequent U, there is a cut only topological play

$$(U : \Gamma \triangleright \Delta) \xrightarrow{c_U} (\Gamma \vdash \Delta),$$

Thus, for any (winning) strategy S on U, there is a (winning) strategy $ce_{c_U}(S)$ on $\Gamma \vdash \Delta$. In order to obtain categories of strategies closer to usual game semantics, we might want to quotient our categories Strat_0 and WStrat_0 by decreeing that two strategies (U, S) and (V, T) from A to B are equivalent when $ce_{c_U}(S) = ce_{c_V}(T)$. Alternatively, we could take morphisms $A \to B$ to be (winning) strategies on the sequent $A \vdash B$, and composition to be defined by amalgamation followed by descent along the cut play, say, from (15) to (14) above. However, this appears trickier than expected, specifically w.r.t. associativity of composition in the obtained candidate categories, and we leave it for further work.

5. Related and further work

The game in this paper is almost the same as in an earlier talk (Hirschowitz et al. 2007), with a few evolutions. The development is very different: in Hirschowitz et al. (2007), we were concerned with making plays a stack, which here is avoided by passing directly to strategies. The notion of strategy we adopt here is radically new – Hirschowitz et al. (2007) used sets of proper plays. Finally, we provide correctness and completeness results which were not in Hirschowitz et al. (2007).

Delande and Miller (2008) investigate a closely related game, with analogous correctness and completeness results. Their approach first technically differs in the way the game ends, and in the definition of won positions. More importantly, their game does not feature any cut move, so they do not deal with cut elimination in any sense. Finally, they do not use topological methods at all.

Melliès (2004) and subsequent papers propose notions of games where plays should be considered up to permutation of certain moves. Our game certainly has an asynchronous flavor in this sense, where permutations arise directly from the topology. However, a formal connection remains to be established.

Most striking are probably the similarities with Girard's (2001) ludics, from which we gratefully acknowledge inspiration. A first difference is technical: Girard does not use topological methods at all, maybe because ludics are restricted to a very particular form of graphs. Also, our game is closer to MALL sequent calculus than ludics, e.g., it does not feature the *daimon* move of ludics. Furthermore, our edges are labeled with formulae, which fixes their behavior – there is exactly one strategy per edge. A key ingredient to Girard's approach is to avoid labels, and instead say that a strategy follows a typing (i.e., a labeling of edges with formulae) when its restriction to each edge behaves accordingly. Adapting our game to this approach is left for further work. Finally, ludics' strategies are still defined as sets of plays, i.e., non locally.

More intrinsically to our game, there are a number of possible directions for improvement. First, as evoked in Section 4.4, our game has to be adapted to fit into a category of strategies. Furthermore, one might want to tighten the connection between strategies and proofs, e.g., towards a full completeness result. Finally, we will try to extend our game to exponentials, and (at least first-order) quantification. This promises to be more difficult, particularly w.r.t. noetherianness.

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